

Bethe-Salpeter approach for relativistic positronium in a strong magnetic field

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Abstract

We study the electron-positron system in a strong magnetic field using the differential Bethe-Salpeter equation in the ladder approximation. We derive the fully relativistic two-dimensional form that the four-dimensional Bethe-Salpeter equation takes in the limit of asymptotically strong constant and homogeneous magnetic field. An ultimate value for the magnetic field is determined, which provides the full compensation of the positronium rest mass by the binding energy in the maximum symmetry state and vanishing of the energy gap separating the electron-positron system from the vacuum. The compensation becomes possible owing to the falling to the center phenomenon that occurs in a strong magnetic field because of the dimensional reduction. The solution of the Bethe-Salpeter equation corresponding to the vanishing energy-momentum of the electron-positron system is obtained.

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I. INTRODUCTION

It is well known that the structure of atoms (positronium included) is drastically modified by a magnetic field \mathbf{B} if the field strength $B = |\mathbf{B}|$ exceeds the characteristic atomic value $B_a = m^2 e^3 c / \hbar^3 \simeq 2.35 \times 10^9$ G [1, 2]. In a strong magnetic field ($B \gg B_a$) the usual perturbative treatment of the magnetic effects (such as Zeeman splitting of atomic energy levels) is not applicable, and instead, the Coulomb forces act as a perturbation to the magnetic forces. For positronium in such a field the characteristic size of the electron and positron across \mathbf{B} is the Larmour radius

$$L_B = (eB)^{-1/2} = a_0 (B_a/B)^{1/2} \quad (1)$$

and decreases with increase of the field strength, where a_0 is the Bohr radius, $a_0 = (\alpha m)^{-1}$, $\alpha = 1/137$. Henceforth, we set $\hbar = c = 1$ and refer to the Heaviside-Lorentz system of units, where the fine structure constant $\alpha = e^2/4\pi$.

The properties of positronium in a strong magnetic field ($B \gg B_a$) are interesting for astrophysics because such fields are observed now for several kinds of astronomical compact objects (pulsars, powerful X-ray sources, soft gamma-ray repeaters, etc.). Besides, some of these objects are the sources of electron-positron pairs produced in their vicinities by various mechanisms [3]. At least a part of these pairs may be bound. For instance, at the surface of radio pulsars identified with rotation-powered neutron stars the field strength is in the range from $\sim 10^9$ G to $\sim 10^{14}$ G [4]. A common point of all available models of pulsars is that electron-positron pairs dominate in the magnetosphere plasma [5]. These are formed by the single-photon production process in a strong magnetic field, $\gamma + B \rightarrow e^+ + e^- + B$. If the field strength is higher than $\sim 4 \times 10^{12}$ G the pairs created are mainly bound [6]. Much more intense magnetic fields have been conjectured to be involved in several astrophysical phenomena. For instance, superconductive cosmic strings, if they exist, may have magnetic fields up to $\sim 10^{47} - 10^{48}$ G in their vicinities [7]. Electron-positron pairs may be produced near such strings [8].

In magnetic fields larger than B_a , the Coulomb force becomes more effective in binding the positronium because the charged constituents are confined to the lowest Landau level and hence to a narrow region stretching along the magnetic field ($L_B \ll a_0$). Notwithstanding this effect, the binding energy of positronium ΔE is still very small in comparison with the rest mass, $\Delta E \ll 2m$, even for the fields larger than Schwinger's critical value $B_0 =$

$m^2/e \simeq 4.4 \times 10^{13}$ G, , i.e., the positronium remains an internally nonrelativistic system. The binding energy of the ground state, as calculated nonrelativistically,

$$\Delta E \simeq \frac{m\alpha^2}{4} \left(\ln \frac{B}{B_0} \right)^2, \quad (2)$$

increases with increase of B , and the relativistic effects, for extremely huge fields, should be expected to become essential. The unrestricted growth of the binding energy (2) with the magnetic field is a manifestation of the fact that the Coulomb attraction force becomes supercritical in the one-dimensional Schrödinger equation, to which the nonrelativistic problem is reduced in the high-field limit [1], and the falling-to-the-center phenomenon occurs in the limit $B = \infty$.

Relativistic properties of positronium in a strong magnetic field were studied basing on the Bethe-Salpeter equation [9, 10]. The nontrivial energy dependence upon the transversal (pseudo)momentum component of the center-of-mass was found in [10, 11]. Although the Bethe-Salpeter equation is fully relativistic, it was used within the customary "equal-time" approximation that disregards the retardation effects, so that the relative motion of the electron and positron is treated in a nonrelativistic way. In this way the behavior (2) is reproduced for the ground state [9] - [11]. A completely relativistic solution for positronium in a strong magnetic field remains unknown. In this paper we study the positronium in an asymptotically strong magnetic field with not only the center-of-mass motion considered relativistically, but also the relative motion of its constituents. We point the ultimate value of the magnetic field guaranteeing such deepening of the positronium energy level that is sufficient to compensate for the whole rest mass $2m$ of it.

To this end, in Section II we derive the fully relativistic - in two-dimensional Minkowsky space - form that the differential Bethe-Salpeter equation in the ladder approximation takes for the positronium when the magnetic field tends to infinity. This equation is efficient already for $B \gg B_a$. We also include a moderate external electric field parallel to \mathbf{B} into this equation. In Section III the ultra-relativistic solution of maximum symmetry is found to the equation derived in Section II corresponding to the vanishing total energy-momentum of the positronium. The falling-to-the-center phenomenon [12] characteristic of the two-dimensional equation of Section II for every positive value of the fine structure constant [13] is exploited for establishing the possibility that the zero energy point may belong to the spectrum, provided the magnetic field is sufficiently large. The origin of the falling to the

center is in the ultraviolet singularity of the photon propagator. The effects of the mass radiative corrections and of the vacuum polarization are also considered. In concluding Section IV the results are summarized.

II. TWO-DIMENSIONAL BETHE-SALPETER EQUATION FOR POSITRONIUM IN AN ASYMPTOTICALLY STRONG MAGNETIC FIELD

The view [1] that charged particles in a strong constant magnetic field are confined to the lowest Landau level and behave effectively as if they possess only one spacial degree of freedom - the one along the magnetic field - is widely accepted. Moreover, a conjecture exists [17] that the Feynman rules in the high magnetic field limit may be directly served by two-dimensional (one space + one time) form of electron propagators. As applied to the Bethe-Salpeter equation, the dimensional reduction in high magnetic field was considered in [9, 10]. In these references the well-known simultaneous approximation to the Bethe-Salpeter equation taken in the integral form was exploited, appropriate for nonrelativistic treatment of the relative motion of the two charged particles. Once we shall in the next Section be interested in the ultrarelativistic regime, we reject from using this approximation, and find it convenient to deal only with the differential form of the Bethe-Salpeter equation.

The electron-positron bound state is described by the Bethe-Salpeter amplitude (wave function) $\chi_{\lambda,\beta}(x^e, x^p)$ subject to the fully relativistic equation (e.g., [18]), which in the ladder approximation in a magnetic field may be written as

$$\begin{aligned} & [\hat{i}\hat{\partial}^e - m + e\hat{A}(x^e)]_{\lambda\beta} [\hat{i}\hat{\partial}^p - m - e\hat{A}(x^p)]_{\mu\nu} \chi_{\beta\nu}(x^e, x^p) \\ & = -i8\pi\alpha D^{ij}(x^e - x^p) [\gamma_i]_{\lambda\beta} [\gamma_j]_{\mu\nu} \chi_{\beta\nu}(x^e, x^p). \end{aligned} \quad (3)$$

Here x^e, x^p are the electron and positron 4-coordinates, $D^{ij}(x^e - x^p)$ is the photon propagator, and we have explicitly written the spinor indices $\lambda, \beta, \mu, \nu = 1, 2, 3, 4$. The metrics in the Minkowsky space is $\text{diag } g_{ij} = (1, -1, -1, -1)$, $i, j = 0, 1, 2, 3$. The derivatives

$$\hat{\partial} = \partial^j \gamma_j = \partial^0 \gamma_0 + \partial^k \gamma_k = \gamma_0 \frac{\partial}{\partial x_0} + \gamma_k \frac{\partial}{\partial x_k}, \quad k = 1, 2, 3, \quad (4)$$

act on x^e or x^p as indicated by the superscripts, and $\hat{A} = A_0 \gamma_0 - A_k \gamma_k$.

We consider the ladder approximation with the photon propagator taken in the Feynman gauge. With other gauges this approximation corresponds to summation of diagrams other

than the ladder ones in agreement with the well-known fact that the ladder approximation is not gauge-invariant.

We refer to, if needed, the so called spinor representation of the Dirac γ -matrices in the block form

$$\gamma_0 = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix}, \quad \gamma_k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad (5)$$

σ_k are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (6)$$

m is the electron mass, e the absolute value of its charge, $e = 2\sqrt{\pi\alpha}$. The vector potential of the constant and homogeneous magnetic field B , directed along the axis 3 ($B_3 = B$, $B_{1,2} = 0$), is chosen in the asymmetric gauge

$$A_1(x) = -Bx_2, \quad A_{0,2,3}(x) = 0. \quad (7)$$

With this choice, the translational invariance along the directions 0,1,3 holds.

Solutions to Eq. (3) may be represented in the form

$$\chi(x^e, x^p) = \eta(x_0^e - x_0^p, x_3^e - x_3^p, x_{1,2}^e, x_{1,2}^p) \exp\left\{\frac{i}{2}[P_0(x_0^e + x_0^p) - P_3(x_3^e + x_3^p)]\right\}, \quad (8)$$

where $P_{0,3}$ are the center-of-mass 4-momentum components of the longitudinal motion, that express the translational invariance along the longitudinal directions (0,3). Denoting the differences $x_0^e - x_0^p = t$, $x_3^e - x_3^p = z$, from Eqs. (3) and (8) we obtain

$$\begin{aligned} & \left[i\hat{\partial}_{\parallel} - \frac{\hat{P}_{\parallel}}{2} - m + i\hat{\partial}_{\perp} - e\gamma_1 A_1(x_2^e) \right]_{\lambda\beta} \left[-i\hat{\partial}_{\parallel} - \frac{\hat{P}_{\parallel}}{2} - m + i\hat{\partial}_{\perp} + e\gamma_1 A_1(x_2^p) \right]_{\mu\nu} \\ & \times [\eta(t, z, x_{\perp}^{e,p})]_{\beta\nu} = -i8\pi\alpha D_{ij}(t, z, x_{1,2}^e - x_{1,2}^p) [\gamma_i]_{\lambda\beta} [\gamma_j]_{\mu\nu} [\eta(t, z, x_{\perp}^{e,p})]_{\beta\nu}, \end{aligned} \quad (9)$$

where $x_{\perp} = (x_1, x_2)$, $\hat{\partial}_{\perp} = \gamma_1 \frac{\partial}{\partial x_1} + \gamma_2 \frac{\partial}{\partial x_2}$, $\hat{\partial}_{\parallel} = \frac{\partial}{\partial t} \gamma_0 + \frac{\partial}{\partial z} \gamma_3$, and $\hat{P}_{\parallel} = P_0 \gamma_0 - P_3 \gamma_3$.

A. Fourier-Ritus Expansion in eigenfunctions of the transversal motion

Expand the dependence of solution of Eq. (9) on the transversal degrees of freedom into the series over the (complete set of) Ritus [19] matrix eigenfunctions $E_h(x_2)$

$$[\eta(t, z, x_{\perp}^{e,p})]_{\mu\nu} = \sum_{h^e h^p} e^{ip_1^e x_1^e} [E_{h^e}^e(x_2^e)]_{\mu}^{\lambda^e} [E_{h^p}^p(x_2^p)]_{\nu}^{\lambda^p} e^{ip_1^p x_1^p} [\eta_{h^e h^p}(t, z)]_{\lambda^e \lambda^p}. \quad (10)$$

Here $\eta_{h^e h^p}(t, z)$ denote *unknown* functions that depend on the differences of the longitudinal variables, while the Ritus matrix functions $e^{ip_1 x_1} E_h(x_2)$ depend on the individual coordinates $x_{1,2}^{e,p}$ transversal to the field. The Ritus matrix functions and the unknown functions $\eta_{h^e h^p}(t, z)$ are labelled by two pairs h^e, h^p of quantum numbers $h = (k, p_1)$, each pair relating to one out of the two particles in a magnetic field. The Landau quantum number k runs all nonnegative integers, $k = 0, 1, 2, 3, \dots$, while p_1 is the particle momentum component along the transversal axis 1. Recall that the potential $A_\mu(x)$ (7) does not depend on x_1 , so that p_1 does conserve. This quantum number is connected with the orbit center coordinate \tilde{x}_2 along the axis 2 [12], $p_1 = -\tilde{x}_2 eB$.

The matrix functions $e^{ip_1 x_1} E_h^{e,p}(x_2)$ for transverse motion in the magnetic field (7), relating in (10) to electrons (e) and positrons (p), are 4×4 matrices, formed, in the spinor representation, by four eigen-bispinors of the operator $(-i\hat{\partial}_\perp \pm e\hat{A})^2$

$$(-i\hat{\partial}_\perp \pm e\hat{A})_{\mu\nu}^2 e^{ip_1 x_1} [E_h^{e,p}(x_2)]_\nu^{(\sigma,\gamma)} = -2eBk e^{ip_1 x_1} [E_h^{e,p}(x_2)]_\mu^{(\sigma,\gamma)}, \quad (11)$$

placed, as columns, side by side [19]. Here the upper and lower signs relate to electron and positron, respectively, while $\sigma = \pm 1$ and $\gamma = \pm 1$ are eigenvalues of the operators

$$\Sigma_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad -i\gamma_5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \quad (12)$$

diagonal in the spinor representation, to which the same 4-spinors are eigen-bispinors [20]

$$-i\gamma_5 E_h^{(\sigma,\gamma)} = \gamma E_h^{(\sigma,\gamma)}, \quad \Sigma_3 E_h^{(\sigma,\gamma)} = \sigma E_h^{(\sigma,\gamma)}. \quad (13)$$

The couple of indices $\lambda = (\sigma, \gamma)$ is united into one index λ in the expansion (10), $\lambda = 1, 2, 3, 4$ according to the convention: $(+1, -1) = 1$, $(-1, -1) = 2$, $(+1, +1) = 3$, $(-1, +1) = 4$. With this convention, the set of 4-spinors $[E_h(x_2)]_\mu^{(\sigma,\gamma)} = E_h(x_2)_\mu^\lambda$ can be dealt with as a 4×4 matrix, the united index λ spanning a matrix space, where the usual algebra of γ -matrices may act. Correspondingly, in (10) the unknown function $[\eta_{h^e h^p}(t, z)]_{\lambda^e \lambda^p}$ is a matrix in the same space, and contracts with the Ritus matrix function.

Following [19], the matrix functions in expansion (10) can be written in the block form

as diagonal matrices

$$\begin{aligned} e^{ip_1 x_1} E_h^{e,p}(x_2) &= \begin{pmatrix} a^{e,p}(h; x_{1,2}) & 0 \\ 0 & a^{e,p}(h; x_{1,2}) \end{pmatrix}, \\ a^{e,p}(h; x_{1,2}) &= \begin{pmatrix} a_{+1}^{e,p}(h; x_{1,2}) & 0 \\ 0 & a_{-1}^{e,p}(h; x_{1,2}) \end{pmatrix}. \end{aligned} \quad (14)$$

Here $a_\sigma^{e,p}(h; x_{1,2})$ are eigenfunctions of the two (for each sign \pm) operators $[((-i\partial_\perp)_\lambda \pm eA_\lambda)^2 \mp \sigma eB]$ (we denote $(\partial_\perp)_\lambda = \partial/\partial x_\lambda$, $\lambda = 1, 2$), labelled by the two values $\sigma = 1, -1$,

$$[((-i\partial_\perp)_\lambda \pm eA_\lambda)^2 \mp \sigma eB] a_\sigma^{e,p}(h; x_{1,2}) = 2eBk a_\sigma^{e,p}(h; x_{1,2}), \quad (15)$$

namely, (we omit the subscript "1" by p_1 in what follows)

$$a_\sigma^{e,p}(h; x_{1,2}) = e^{ipx_1} U_{k+\frac{\pm\sigma-1}{2}} \left[\sqrt{eB} \left(x_2 \pm \frac{p}{eB} \right) \right], \quad k = 0, 1, 2, \dots, \quad (16)$$

with

$$U_n(\xi) = \exp \left\{ -\frac{\xi^2}{2} \right\} (2^n n! \sqrt{\pi})^{-\frac{1}{2}} H_n(\xi) \quad (17)$$

being the normalized Hermite functions ($H_n(\xi)$ are the Hermite polynomials). Eqs. (15) are the same as (11) due to the relation

$$(\hat{i}\hat{\partial}_\perp \mp e\hat{A})^2 = -[(i\partial_\perp)_\lambda \mp eA_\lambda]^2 \pm eB\Sigma_3 \quad (18)$$

and to Eq. (13). Simultaneously, the matrix functions (14) are eigenfunctions to the operator $-i\partial_1$ that commutes with Σ_3 and γ_5 (12), and with $(\hat{i}\hat{\partial}_\perp \mp e\hat{A})_{\mu\nu}^2$. The corresponding eigenvalue p_1 does not, however, appear in the r.h. side of (15) due to the well-known degeneracy of electron spectrum in a constant magnetic field.

The orthonormality relation for the Hermite functions

$$\int_{-\infty}^{\infty} U_n(\xi) U_{n'}(\xi) d\xi = \delta_{nn'}. \quad (19)$$

implies the orthogonality of the Ritus matrix eigenfunctions in the form

$$\sqrt{eB} \int E_h^*(x_2)_\mu^\lambda E_{h'}(x_2)_\mu^\lambda dx_2 = \delta_{kk'} \delta_{\lambda\lambda'}. \quad (20)$$

As a matter of fact, the matrix functions $E_h(x_2)$ are real, and we henceforth omit the complex conjugation sign ”*”.

The matrix functions $e^{ipx_1} E_h^{e,p}(x_2)$ (14) commute with the longitudinal part $\pm i\hat{\partial}_{\parallel} - \hat{P}_{\parallel}/2 - m$ of the Dirac operator in (9), owing to the commutativity property

$$[E_h(x_2), \gamma_{0,3}]_- = 0, \quad (21)$$

and are [19], in a sense, matrix eigenfunctions of the transversal part of the Dirac operator (not only of its square (11))

$$(i\hat{\partial}_{\perp} \mp e\hat{A})e^{ipx_1} E_h^{e,p}(x_2) = \pm\sqrt{2eBk} e^{ipx_1} E_h^{e,p}(x_2)\gamma_1. \quad (22)$$

The Landau quantum number k appears here as a ”universal eigenvalue” thanks to the mechanism, easy to trace, according to which the differential operator in the left-hand side of Eq. (22) acts as a lowering or rising operator on the functions (17), whereas the matrix σ_2 , involved in γ_2 , interchanges the places the functions U_k, U_{k-1} occupy in the columns. Contrary to relations, which explicitly include the variable σ , whose value forms the number of the corresponding column, relations (11), (22), (21), and the first relation in (13) are covariant with respect to passing to other representation of γ -matrices, where the matrix $E_h(x_2)$ may become non-diagonal.

B. Equation for the Fourier-Ritus transform of the Bethe-Salpeter amplitude

Now we are in a position to use expansion (10) in Eq. (9). We left multiply it by $(2\pi)^{-2}eB e^{-i\bar{p}^e x_1^e} E_{\bar{h}}^e(x_2^e)e^{-i\bar{p}^p x_1^p} E_{\bar{h}^p}^p(x_2^p)$, then integrate over $d^2x_{1,2}^e d^2x_{1,2}^p$. After using (22) and (21), and exploiting the orthonormality relation (20) for the summation over the quantum numbers $h^{e,p} = (k^{e,p}, p_1^{e,p})$, the following expression is obtained for the left-hand side of the Fourier-Ritus-transformed Eq. (9):

$$\left[i\hat{\partial}_{\parallel} - \frac{\hat{P}_{\parallel}}{2} - m - \gamma_1\sqrt{2eBk^e} \right]_{\lambda\lambda^e} \left[-i\hat{\partial}_{\parallel} - \frac{\hat{P}_{\parallel}}{2} - m + \gamma_1\sqrt{2eBk^p} \right]_{\mu\lambda^p} [\eta_{h^e h^p}(t, z)]_{\lambda^e \lambda^p}. \quad (23)$$

We omitted the bars over the quantum numbers.

Taking the expression

$$D_{ij}(t, z, x_{1,2}^e - x_{1,2}^p) = \frac{g_{ij}}{i4\pi^2} [t^2 - z^2 - (x_1^e - x_1^p)^2 - (x_2^e - x_2^p)^2]^{-1}, \quad (24)$$

for the photon propagator in the Feynman gauge, we may then write the right-hand side of Ritus-transformed Eq. (9) as

$$\frac{\alpha}{2\pi^3} \int dp^e dp^p \sum_{k^e k^p} g_{ij} \int [E_{\bar{h}^e}^e(x_2^e) \gamma_i E_{h^e}^e(x_2^e)]_{\lambda\lambda^e} [E_{\bar{h}^p}^p(x_2^p) \gamma_j E_{h^p}^p(x_2^p)]_{\mu\lambda^p} [\eta_{h^e h^p}(t, z)]_{\lambda^e \lambda^p} \times \frac{e^{i(p^e - \bar{p}^e)x_1} e^{i(p^p - \bar{p}^p)x_1} eB d^2 x_{1,2}^e d^2 x_{1,2}^p}{z^2 + (x_1^e - x_1^p)^2 + (x_2^e - x_2^p)^2 - t^2}, \quad (25)$$

Integrating explicitly the exponentials in (25) over the variable $X = (x_1^e + x_1^p)/2$, we obtain the following expression:

$$\frac{\alpha}{\pi^2} \int dp dP_1 \delta(\bar{P}_1 - P_1) \sum_{k^e k^p} g_{ij} \int [E_{\bar{h}^e}^e(x_2^e) \gamma_i E_{h^e}^e(x_2^e)]_{\lambda\lambda^e} [E_{\bar{h}^p}^p(x_2^p) \gamma_j E_{h^p}^p(x_2^p)]_{\mu\lambda^p} \times [\eta_{h^e h^p}(t, z)]_{\lambda^e \lambda^p} \frac{\exp(ix(\bar{p} - p))dx}{z^2 + x^2 + (x_2^e - x_2^p)^2 - t^2} eB dx_2^e dx_2^p, \quad (26)$$

where the new integration variables $x = x_1^e - x_1^p$, $P_1 = p^e + p^p$, $p = (p^e - p^p)/2$ and the new definitions $\bar{P}_1 = \bar{p}^e + \bar{p}^p$, $\bar{p} = (\bar{p}^e - \bar{p}^p)/2$ have been introduced. The pairs of quantum numbers in (26) are

$$\bar{h}^{e,p} = (\bar{k}^{e,p}, \frac{\bar{P}_1}{2} \pm \bar{p}), \quad h^{e,p} = (k^{e,p}, \frac{P_1}{2} \pm p). \quad (27)$$

Hence the arguments of the functions (16) in (26) are:

$$\sqrt{eB} \left(x_2^e + \frac{\bar{P}_1 + 2\bar{p}}{2eB} \right), \quad \sqrt{eB} \left(x_2^e + \frac{P_1 + 2p}{2eB} \right), \quad \sqrt{eB} \left(x_2^p - \frac{\bar{P}_1 - 2\bar{p}}{2eB} \right), \quad \sqrt{eB} \left(x_2^p - \frac{P_1 - 2p}{2eB} \right), \quad (28)$$

successively as the functions $E_h(x_{1,2})$ appear in (26) from left to right. After fulfilling the integration over dP_1 with the use of the δ -function, introduce the new integration variable $q = p - \bar{p}$ instead of p , and the integration variables $\bar{x}_2^e = x_2^e + (\bar{P}_1 + 2\bar{p})/2eB$, $\bar{x}_2^p = x_2^p - (\bar{P}_1 - 2\bar{p})/2eB$ instead of x_2^e and x_2^p . Then (26) may be written as

$$\frac{\alpha}{\pi^2} \int dq \sum_{k^e k^p} g_{ij} \int [E_{\bar{h}^e}^e(x_2^e) \gamma_i E_{h^e}^e(x_2^e)]_{\lambda\lambda^e} [E_{\bar{h}^p}^p(x_2^p) \gamma_j E_{h^p}^p(x_2^p)]_{\mu\lambda^p} [\eta_{h^e h^p}(t, z)]_{\lambda^e \lambda^p} \times \int \frac{\exp(-ixq) dx eB d\bar{x}_2^e d\bar{x}_2^p}{z^2 + x^2 + \left(\bar{x}_2^e - \bar{x}_2^p - \frac{\bar{P}_1 - q}{eB} \right)^2 - t^2}. \quad (29)$$

Now the pairs of quantum numbers in (29) are

$$\bar{h}^{e,p} = (\bar{k}^{e,p}, \frac{\bar{P}_1}{2} \pm \bar{p}), \quad h^{e,p} = (k^{e,p}, \frac{\bar{P}_1}{2} \pm q \pm \bar{p}). \quad (30)$$

Hence the arguments of the functions (16) in (29) from left to right are

$$\sqrt{eB} \bar{x}_2^e, \quad \sqrt{eB} \left(\bar{x}_2^e + \frac{q}{eB} \right), \quad \sqrt{eB} \left(\bar{x}_2^p - \frac{q}{eB} \right), \quad \left(\sqrt{eB} \bar{x}_2^p \right). \quad (31)$$

C. Adiabatic approximation

Now we aim at passing to the large magnetic field regime in the Bethe-Salpeter equation, with (23) as the left-hand side and (29) as the right-hand side. Define the dimensionless integration variables $w = x\sqrt{eB}$, $q' = q/\sqrt{eB}$, $\xi^{e,p} = \bar{x}_2^{e,p}\sqrt{eB}$ in function (29). Then it takes the form

$$\frac{\alpha}{\pi^2} \int dq' \sum_{k^e k^p} g_{ij} \int [E_{\bar{h}^e}^e(x_2^e) \gamma_i E_{h^e}^e(x_2^e)]_{\lambda\lambda^e} [E_{\bar{h}^p}^p(x_2^p) \gamma_j E_{h^p}^p(x_2^p)]_{\mu\lambda^p} [\eta_{h^e h^p}(t, z)]_{\lambda^e \lambda^p} \times \int \frac{\exp(-i w q') dw d\xi^e d\xi^p}{z^2 + \frac{w^2}{eB} + \frac{1}{eB} \left(\xi^e - \xi^p - \frac{\bar{P}_1}{\sqrt{eB}} - q' \right)^2 - t^2}. \quad (32)$$

The pairs of quantum numbers in (32) are

$$\bar{h}^{e,p} = (\bar{k}^{e,p}, \frac{\bar{P}_1}{2} \pm \bar{p}), \quad h^{e,p} = (k^{e,p}, \frac{\bar{P}_1}{2} \pm q' \sqrt{eB} \pm \bar{p}). \quad (33)$$

The arguments of the functions (16) in (32) from left to right are

$$\xi^e, \quad \xi^e + q', \quad \xi^p - q', \quad \xi^p. \quad (34)$$

When considering the large field behavior we admit for completeness that the difference between the centers of orbits along the axis 2 $\tilde{x}_2^e - \tilde{x}_2^p = -\frac{\bar{P}_1}{eB}$ may be kept finite, in other words that the transversal momentum \bar{P}_1 grows linearly with the field. We shall see that that big transversal momenta do not contradict dimensional compactification, but produce an extra regularization of the light-cone singularity.

In the region, where the 2-interval $(z^2 - t^2)^{1/2}$ essentially exceeds the Larmour radius $L_B = (eB)^{-1/2}$,

$$z^2 - t^2 \gg L_B^2, \quad (35)$$

one may neglect the dependence on the integration variables w and later on $\xi^{e,p}$ in the denominator. Integration over w produces $2\pi\delta(q')$, which annihilates the dependence on q' in the arguments (31) of the Hermite functions, and they all equalize.

Let us depict this mechanism in more detail. Fulfill explicitly the integration over dw in (32):

$$\int \frac{\exp(-i w q') dw}{z^2 - t^2 + \frac{w^2}{eB} + \frac{A^2}{eB}} = \frac{\sqrt{eB}\pi}{\sqrt{z^2 - t^2 + \frac{A^2}{eB}}} \times \left[\theta(q') \exp[-q' \sqrt{eB}(z^2 - t^2) + A^2] + \theta(-q') \exp[q' \sqrt{eB}(z^2 - t^2) + A^2] \right], \quad (36)$$

where

$$A^2 = \left(\xi^e - \xi^p - \frac{\bar{P}_1}{\sqrt{eB}} - q' \right)^2 \quad (37)$$

and $\theta(q')$ is the step function,

$$\theta(q') = \begin{cases} 1 & \text{when } q' > 0, \\ \frac{1}{2} & \text{when } q' = 0, \\ 0 & \text{when } q' < 0. \end{cases} \quad (38)$$

Due to the decreasing exponential in (17) the variables $\xi^{e,p}$ do not exceed unity in the order of magnitude and can be neglected as compared to $\frac{\bar{P}_1}{\sqrt{eB}}$ in (37). Unless q' is large it may be neglected as compared to the same term in (37), too. Then $A^2 = \frac{\bar{P}_1^2}{eB}$, and after (36) is substituted in (32) and integrated over dq' the contribution comes only from the integration within the shrinking region $|q'| < (eB[z^2 - t^2 + \frac{\bar{P}_1^2}{(eB)^2}])^{-\frac{1}{2}}$. Then q' can be also neglected in the arguments (34). If, contrary to the previous assumption, we admit that $|q'|$ is of the order of $\frac{\bar{P}_1}{\sqrt{eB}} \sim \sqrt{eB}$ we see that the exponentials in (36) fast decrease with the growth of the magnetic field as $\exp[-eB(z^2 - t^2)]$, and therefore such values of $|q'|$ do not contribute to the integration. If we admit, last, that $|q'| \gg |\frac{\bar{P}_1}{\sqrt{eB}}|$, we find that the contribution $\exp[-|q'|\sqrt{eB(z^2 - t^2) + (q')^2}]$ from the integration over such values is still smaller. Thus, we have justified the possibility to omit the dependence on q' in (37) and in (34), and also on $\xi^{e,p}$ in (37). Now we can perform the integration over dq' to obtain the following expression for (32):

$$\frac{2\alpha\pi^{-1}}{z^2 + \frac{P_1^2}{(eB)^2} - t^2} \sum_{k^e k^p} g_{ii} \int [E_h^e(x_2^e) \gamma_i E_{h^e}^e(x_2^e)]_{\lambda\lambda^e} d\xi^e \int [E_h^p(x_2^p) \gamma_i E_{h^p}^p(x_2^p)]_{\mu\lambda^p} d\xi^p [\eta_{h^e h^p}(t, z)]_{\lambda^e \lambda^p}. \quad (39)$$

It remains yet to argue that the limit (39) is valid also when the term $\frac{\bar{P}_1}{eB}$ is not kept. In this case we no longer can disregard q' inside A^2 when q' is less than or of the order of unity. But we can disregard A^2 as compared with $eB(z^2 - t^2)$ to make sure that the integration over dq' is restricted to the region close to zero $|q'| < [eB(z^2 - t^2)]^{-1/2}$ and hence set $q' = 0$ in (34). The contribution of large q' is small as before.

The integration over $\xi^{e,p}$ of the terms with $i = 0, 3$ in (39) yields the Kronecker deltas $\delta_{k^e \bar{k}^e} \delta_{k^p \bar{k}^p}$ due to the orthonormality (19) of the Hermite functions thanks to the commutativity (21) of the Ritus matrix functions (14) with γ_0 and γ_3 . On the contrary, γ_1, γ_2

do not commute with (14). This implies the appearance of terms, non-diagonal in Landau quantum numbers, like $\delta_{k^e, \bar{k}^e \pm 1}$ and $\delta_{k^p, \bar{k}^p \pm 1}$, in (32), proportional to ($i = 1, 2$):

$$\begin{aligned} T_{k^e \pm 1, \bar{k}^p \pm 1}^i &= \sum_{k^e k^p} \int [E_{\bar{h}}^e(x_2^e) \gamma_i E_{h^e}^e(x_2^e)]_{\lambda \lambda^e} d\xi^e \int [E_{\bar{h}^p}^p(x_2^p) \gamma_i E_{h^p}^p(x_2^p)]_{\mu \lambda^p} d\xi^p [\eta_{h^e h^p}(t, z)]_{\lambda^e \lambda^p} \\ &= \sum_{k^e k^p} \begin{pmatrix} 0 & -\Delta_{\bar{k}^e k^e}^i \\ \Delta_{\bar{k}^e k^e}^i & 0 \end{pmatrix}_{\lambda \lambda^e} \begin{pmatrix} 0 & -\Delta_{\bar{k}^p k^p}^i \\ \Delta_{\bar{k}^p k^p}^i & 0 \end{pmatrix}_{\mu \lambda^p} [\eta_{h^e h^p}(t, z)]_{\lambda^e \lambda^p}. \end{aligned} \quad (40)$$

Here $x_2^{e,p}$ are expressed in terms of ξ through the chain of the changes of variables made above starting from (25), so that all the arguments of the Hermite functions have become equal to ξ . Besides,

$$h^{e,p} = (k^{e,p}, \bar{p}^{e,p}), \quad \bar{h}^{e,p} = (\bar{k}^{e,p}, \bar{p}^{e,p}), \quad p^e + p^p = P_1. \quad (41)$$

$$\Delta_{\bar{k}k}^i = \int a'(\bar{h}, x_2) \sigma_i a'(h, x_2) d\xi, \quad i = 1, 2, \quad (42)$$

$$\Delta_{\bar{k}k}^{(1)} = \int \begin{pmatrix} 0 & a'_{+1}(\bar{h}, x_2) a'_{-1}(k, x_2) \\ a'_{-1}(\bar{h}, x_2) a'_{+1}(h, x_2) & 0 \end{pmatrix} d\xi = \begin{pmatrix} 0 & \delta_{\bar{k}, k-1} \\ \delta_{\bar{k}, k+1} & 0 \end{pmatrix}, \quad (43)$$

$$\Delta_{\bar{k}k}^{(2)} = i \int \begin{pmatrix} 0 & -a'_{+1}(\bar{h}, x_2) a'_{-1}(k, x_2) \\ a'_{-1}(\bar{h}, x_2) a'_{+1}(h, x_2) & 0 \end{pmatrix} d\xi = i \begin{pmatrix} 0 & -\delta_{\bar{k}, k-1} \\ \delta_{\bar{k}, k+1} & 0 \end{pmatrix}. \quad (44)$$

The prime over a indicates that the exponential $\exp(ipx_1)$ is dropped from the definitions (14) and (16). The non-diagonal Kronecker deltas appeared, because $a'_{\pm 1}(\bar{h}, x_2)$ are multiplied by $a'_{\mp 1}(h, x_2)$ under the action of the $\sigma_{1,2}$ -blocks in $\gamma_{1,2}$ (5). In the final form, the matrices in (40) are

$$\begin{pmatrix} 0 & -\Delta_{\bar{k}k}^i \\ \Delta_{\bar{k}k}^i & 0 \end{pmatrix} = \frac{1}{2} (\gamma_1 (\pm \delta_{\bar{k}, k-1} + \delta_{\bar{k}, k+1}) + i \gamma_2 (\pm \delta_{\bar{k}, k-1} - \delta_{\bar{k}, k+1})), \quad (45)$$

with the upper sign relating to $i = 1$ and the lower one to $i = 2$. Now Eq. (9) acquires the following form:

$$\begin{aligned} & \left[i \hat{\partial}_{\parallel} - \frac{\hat{P}_{\parallel}}{2} - m - \gamma_1 \sqrt{2eBk^e} \right]_{\lambda \lambda^e} \left[-i \hat{\partial}_{\parallel} - \frac{\hat{P}_{\parallel}}{2} - m + \gamma_1 \sqrt{2eBk^p} \right]_{\mu \lambda^p} [\eta_{h^e h^p}(t, z)]_{\lambda^e \lambda^p} \\ &= \frac{2\alpha\pi^{-1}}{z^2 + \frac{P_1^2}{(eB)^2} - t^2} \left(\sum_{i=0,3} g_{ii} [\gamma_i]_{\lambda \lambda^e} [\gamma_i]_{\mu \lambda^p} [\eta_{h^e h^p}(t, z)]_{\lambda^e \lambda^p} - \sum_{i=1,2} T_{k^e \pm 1, k^p \pm 1}^{(i)} \right). \end{aligned} \quad (46)$$

The bars over quantum numbers are omitted. This equation is degenerate with respect to the difference of the electron and positron momentum components $p = (p^e - p^p)/2$ across the magnetic field, but does depend on its transversal center-of-mass momentum $P_1 = (p^e + p^p)$. This dependence is present, however, only for sufficiently large transverse momenta P_1 .

At the present step of adiabatic approximation we have come, for high magnetic field, to the chain of Eqs. (46), in which the unknown function for a given pair of Landau quantum numbers k^e, k^p is tangled with the same function with the Landau quantum numbers both shifted by ± 1 (in contrast to the general case of a moderate magnetic field, where these numbers may be shifted by all positive and negative integers). To be more precise, the chain consists of two mutually disentangled sub-chains. The first one includes all functions with the Landau quantum numbers k^e, k^p both even or both odd, and the second includes their even-odd and odd-even combinations. We discuss the first sub-chain since it contains the lowest function with $k^e = k^p = 0$. We argue now that there exists a solution to the first sub-chain of Eqs. (46), for which all $\eta_{k^e, p_1^e; k^p, p_1^p}(t, z)$ disappear if at least one of the quantum numbers k^e, k^p is different from zero. Indeed, for $k^e = k^p = 0$ Eqs. (46) then reduces to the closed set

$$\begin{aligned} & [i\hat{\partial}_{\parallel} - \frac{\hat{P}_{\parallel}}{2} - m]_{\lambda\lambda^e} [-i\hat{\partial}_{\parallel} - \frac{\hat{P}_{\parallel}}{2} - m]_{\mu\lambda^p} [\eta_{0, p_1^e; 0, p_1^p}(t, z)]_{\lambda^e \lambda^p} \\ &= \frac{2\alpha\pi^{-1}}{z^2 + \frac{P_1^2}{(eB)^2} - t^2} \sum_{i=0,3} g_{ii}[\gamma_i]_{\lambda\lambda^e} [\gamma_i]_{\mu\lambda^p} [\eta_{0, p_1^e; 0, p_1^p}(t, z)]_{\lambda^e \lambda^p}, \quad p_1^e + p_1^p = P_1. \end{aligned} \quad (47)$$

In writing it we have returned to the initial designation of the electron and positron transverse momenta $p_1^{e,p}$. Denote for simplicity $\eta_{k^e k^p} = \eta_{k^e, p_1^e; k^p, p_1^p}(t, z)$. If we consider Eqs. (46) with $k^e = k^p = 1$ for η_{11} we shall have a nonzero contribution in the right-hand side, proportional to η_{00} coming from T_{k^e-1, k^p-1}^i , since the other contributions $\eta_{11}, \eta_{22}, \eta_{20}, \eta_{02}$ are vanishing according to the assumption. As the left-hand side of Eq. (46) now contains a term, infinitely growing with the magnetic field B , it can be only satisfied with the function η_{11} , infinitely diminishing with B in the domain (35) as

$$[\eta_{11}]_{\lambda\mu} = -\frac{1}{2eB} \frac{\alpha\pi^{-1}}{z^2 + \frac{P_1^2}{(eB)^2} - t^2} \sum_{i=0,3} g_{ii}[\gamma_1 \gamma_i]_{\lambda\lambda^e} [\gamma_1 \gamma_i]_{\mu\lambda^p} [\eta_{00}]_{\lambda^e \lambda^p}, \quad (48)$$

in accord with the assumption made. Thus, the assumption that all Bethe-Salpeter amplitudes with nonzero Landau quantum numbers are zero in the large-field case is consistent. We state that a solution to the closed set (47) for $\eta_{0, p_1^e; 0, p_1^p}(t, z)$ with all the other components equal to zero is a solution to the whole chain (46).

The derivation given in this Subsection realizes formally the known heuristic argument that, for high magnetic field, the spacing between Landau levels is very large and hence the particles taken in the lowest Landau state remain in it. Effectively, only the longitudinal degree of freedom survives for large B , the space-time reduction taking place. Eq. (47) is a fully relativistic two-dimensional set of equations with two space-time arguments t and z and two gamma-matrices γ_0 and γ_3 involved. Since, unlike the previous works [9], [10], [11], neither the famous equal-time Ansatz for the Bethe-Salpeter amplitude [18], nor any other assumption concerning the non-relativistic character of the internal motion inside the positronium atom was made, the equation derived is valid for arbitrary strong binding. It will be analyzed for the extreme relativistic case in the next Section.

The two-dimensional equation (47) is valid in the space-like domain (35). It is meaningful provided that its solution is concentrated in this domain. In non-relativistic or semi-relativistic consideration it is often accepted that the wave function is concentrated within the Bohr radius $a_0 = (\alpha m)^{-1} \simeq 0.5 \times 10^{-8}$ cm. It is then estimated that the corresponding analog of asymptotic equation (47) holds true when $a_0 \gg L_B$, i.e. for the magnetic fields larger than $B_a = \alpha^2 m^2 / e \simeq 2.35 \times 10^9$ G. This estimate, however, cannot be universal and may be applicable at the most to the magnetic fields close to the lower bound where the value of the Bohr radius can be borrowed from the theory without the magnetic field. Generally, the question, where the wave function is concentrated, should be answered *a posteriori* by inspecting a solution to Eq. (47). Therefore, one can state, how large the fields should be in order that the asymptotic equation (47) might be trusted, no sooner that its solution is investigated. We shall return to this point when we deal with the ultra-relativistic situation.

Remind that the transverse total momentum component of the positronium system is connected with the separation between the centers of orbits of the electron and positron $P_1/(eB) = \tilde{x}_2^e - \tilde{x}_2^p$ in the transversal plane, so that the "potential" factor in Eq. (47) may be expressed in the following interesting form

$$\frac{\alpha}{(x_0^e - x_0^p)^2 - (x_3^e - x_3^p)^2 - (\tilde{x}_2^e - \tilde{x}_2^p)^2}, \quad (49)$$

(cf the corresponding form of the Coulomb potential in the semi-relativistic treatment of the Bethe-Salpeter equation in [10, 11]- the difference between the potentials in [10, 11] lies within the accuracy of the adiabatic approximation). The appearance of P_1^2 in the potential determines the energy spectrum dependence upon the momentum of motion of the

two-particle system across the magnetic field like in [10], [11], [21].

We shall need Eq. (47) in a more convenient form. First, transcribe it as

$$\begin{aligned} & (\vec{i}\hat{\partial}_{\parallel} - \frac{\hat{P}_{\parallel}}{2} - m) \eta_{0,p_1^e;0,p_1^p}(t, z) (-\vec{i}\hat{\partial}_{\parallel} - \frac{\hat{P}_{\parallel}}{2} - m)^T \\ &= \frac{2\alpha\pi^{-1}}{z^2 + \frac{P_1^2}{(eB)^2} - t^2} \sum_{i=0,3} g_{ii}\gamma_i \eta_{0,p_1^e;0,p_1^p}(t, z) \gamma_i^T. \end{aligned} \quad (50)$$

Here the superscript T denotes the transposition. With the help of the relation $\gamma_i^T = -C^{-1}\gamma_i C$, with C being the charge conjugation matrix, $C^2 = 1$, and the anti-commutation relation $[\gamma_i, \gamma_5]_+ = 0$, $\gamma_5^2 = -1$, we may write for a new Bethe-Salpeter amplitude $\Theta(t, z)$, defined as

$$\Theta(t, z) = \eta_{0,p_1^e;0,p_1^p}(t, z) C \gamma_5, \quad (51)$$

the equation

$$\begin{aligned} & (\vec{i}\hat{\partial}_{\parallel} - \frac{\hat{P}_{\parallel}}{2} - m) \Theta(t, z) (-\vec{i}\hat{\partial}_{\parallel} - \frac{\hat{P}_{\parallel}}{2} - m) \\ &= \frac{2\alpha\pi^{-1}}{z^2 + \frac{P_1^2}{(eB)^2} - t^2} \sum_{i=0,3} g_{ii}\gamma_i \Theta(t, z) \gamma_i. \end{aligned} \quad (52)$$

The unknown function Θ here is a 4×4 matrix, which contains as a matter of fact only four independent components. In order to correspondingly reduce the number of equations in the set (52), one should note that the γ -matrix algebra in two-dimensional space-time should have only four basic elements. In accordance with this fact, only the matrices $\gamma_{0,3}$ are involved in (52). Together with the matrix $\gamma_0\gamma_3$ and the unit matrix I they form the basis, since $\gamma_{0,3} \cdot \gamma_0\gamma_3 = \gamma_{3,0}$, $\gamma_0^2 = -\gamma_3^2 = (\gamma_0\gamma_3)^2 = 1$, $[\gamma_0, \gamma_3]_+ = [\gamma_{0,3}, \gamma_0\gamma_3]_+ = 0$. Using this algebra and the general representation for the solution

$$\Theta = aI + b\gamma_0 + c\gamma_3 + d\gamma_0\gamma_3, \quad (53)$$

one readily obtains a closed set of four first-order differential equations for the four functions a, b, c, d of t and z . The same set will be obtained, if one replaces in Eqs. (52) and (53) the 4×4 matrices by the Pauli matrices (6), subject to the same algebraic relations, according, for instance, to the rule: $\gamma_0 \Rightarrow \sigma_3$, $\gamma_3 \Rightarrow i\sigma_2$, $\gamma_0\gamma_3 \Rightarrow \sigma_1$. Then Eq. (47) becomes a matrix equation

$$\begin{aligned} & (i\vec{\partial}_t \sigma_3 + \vec{\partial}_z \sigma_2 - \frac{P_0}{2} \sigma_3 + \frac{P_3}{2} i\sigma_2 - m) \vartheta(t, z) (-i\vec{\partial}_t \sigma_3 - \vec{\partial}_z \sigma_2 - \frac{P_0}{2} \sigma_3 + \frac{P_3}{2} i\sigma_2 - m) \\ &= \frac{2\alpha\pi^{-1}}{z^2 + \frac{P_1^2 - t^2}{(eB)^2}} [\sigma_3 \vartheta(t, z) \sigma_3 + \sigma_2 \vartheta(t, z) \sigma_2] \end{aligned} \quad (54)$$

for a 2×2 matrix ϑ ,

$$\vartheta = aI + b\sigma_3 + ic\sigma_2 + d\sigma_1. \quad (55)$$

Here I is the 2×2 unit matrix, and functions a, b, c, d are the same as in (53).

D. Including an external electric field

Let us generalize the two-dimensional Bethe-Salpeter equation obtained in the presence of a strong magnetic field by including an external electric field, parallel to it, that is not supposed to be strong, $E \ll B$. To this end we supplement the potential (7) in Eq. (3) by two more nonzero components

$$A_0(x_0, x_3), A_3(x_0, x_3) \neq 0, \quad (56)$$

that carry the electric field - not necessarily constant - directed along the axis 3. We shall use the collective notations $A_{\parallel} = (A_0, A_3)$, $x_{\parallel} = (x_0, x_3)$, $\hat{\partial}_{\parallel}^{e,p} = \partial_0^{e,p}\gamma_0 - \partial_3^{e,p}\gamma_3$, $\hat{A}_{\parallel} = A_0\gamma_0 - A_3\gamma_3$. We shall not exploit now a representation like (8), but deal directly with the Bethe-Salpeter amplitude $\chi(x^e, x^p)$ as a function of the electron and positron coordinates, and with its Fourier-Ritus transform $\chi_{h^{ep}}(x_{\parallel}^e, x_{\parallel}^p)$ connected with $\chi(x_{\parallel}^e, x_{\parallel}^p; x_{\perp}^e, x_{\perp}^p)$ in the same way as (10). In place of Eq. (9) one should write

$$\begin{aligned} & \left[i\hat{\partial}_{\parallel}^e - e\hat{A}_{\parallel}(x_{\parallel}^e) - m + i\hat{\partial}_{\perp}^e - e\gamma_1 A_1(x_2^e) \right]_{\lambda\beta} \left[i\hat{\partial}_{\parallel}^p + e\hat{A}_{\parallel}(x_{\parallel}^p) - m + i\hat{\partial}_{\perp}^p + e\gamma_1 A_1(x_2^p) \right]_{\mu\nu} \\ & \times \left[\chi(x_{\parallel}^{e,p}, x_{\perp}^{e,p}) \right]_{\beta\nu} = -i8\pi\alpha D_{ij}(t, z, x_{1,2}^e - x_{1,2}^p) [\gamma_i]_{\lambda\beta} [\gamma_j]_{\mu\nu} \left[\chi(x_{\parallel}^{e,p}, x_{\perp}^{e,p}) \right]_{\beta\nu}. \end{aligned} \quad (57)$$

Thanks to the commutativity (21) the rest of the procedure of the previous Subsection remains essentially the same, and we come, in place of (47), to the following two-dimensional equation:

$$\begin{aligned} & \left[i\hat{\partial}_{\parallel}^e - e\hat{A}_{\parallel}(x_{\parallel}^e) - m \right]_{\lambda\beta} \left[i\hat{\partial}_{\parallel}^p + e\hat{A}_{\parallel}(x_{\parallel}^p) - m \right]_{\mu\nu} \left[\chi_{0,p_1^e;0,p_1^p}(x_{\parallel}^e, x_{\parallel}^p) \right]_{\beta\nu} \\ & = \frac{2\alpha\pi^{-1}}{z^2 + \frac{P_1^2}{(eB)^2} - t^2} \sum_{i=0,3} g_{ii} [\gamma_i]_{\lambda\beta} [\gamma_i]_{\mu\nu} \left[\chi_{0,p_1^e;0,p_1^p}(x_{\parallel}^e, x_{\parallel}^p) \right]_{\beta\nu}, \end{aligned} \quad (58)$$

for a positronium atom in a strong magnetic field placed in a moderate electric field, parallel to the magnetic one. In order to apply this equation to a system of two different oppositely charged particles interacting with each other through the photon exchange and placed into

the combination of a strong magnetic and an electric field in the same direction, say a relativistic hydrogen atom, one should only distinguish the two masses in the first and second square brackets in the left-hand side.

III. ULTRA-RELATIVISTIC REGIME IN A MAGNETIC FIELD

In the ultra-relativistic limit, where the positronium mass is completely compensated by the mass defect, $P_0 = 0$, for the positronium at rest along the direction of the magnetic field $P_3 = 0$, the most general relativistic-covariant form of the solution (53) is

$$\Theta = I\Phi + \hat{\partial}_{\parallel}\Phi_2 + \gamma_0\gamma_3\Phi_3. \quad (59)$$

The point is that $\gamma_0\gamma_3$ is invariant under the Lorentz rotations in the plane (t, z) . Substituting this into (52) with $P_0 = P_3 = 0$ we get a separate equation for the singlet component of (59)

$$(-\square_2 + m^2)\Phi(t, z) = \frac{4\alpha\pi^{-1}\Phi(t, z)}{z^2 + \frac{P_1^2}{(eB)^2} - t^2} \quad (60)$$

and the set of equations

$$\begin{aligned} (\square_2 + m^2)\Phi_3(t, z) &= -\frac{4\alpha\pi^{-1}\Phi_3(t, z)}{z^2 + \frac{P_1^2}{(eB)^2} - t^2}, \\ (-\square_2 + m^2)\partial_t\Phi_2 + 2mi\partial_z\Phi_3 &= 0, \\ (-\square_2 + m^2)\partial_z\Phi_2 + 2mi\partial_t\Phi_3 &= 0 \end{aligned} \quad (61)$$

for the other two components. Here $\square_2 = -\partial^2/\partial t^2 + \partial^2/\partial z^2$ is the Laplace operator in two dimensions. Note the "tachyonic" sign in front of it in the first equation (61).

Let us differentiate the second equation in (61) over z and the third one over t and subtract the results from each other. In this way we get that $\square_2\Phi_3 = 0$. This, however, contradicts the first equation in (61) if $\Phi_3 \neq 0$. Therefore, only $\Phi_3 = 0$ is possible. Then, the two second equations in (61) are satisfied, provided that $(-\square_2 + m^2)\Phi_2 = 0$. We shall concentrate in Eq. (60) in what follows.

The longitudinal momentum along x_1 , or the distance between the orbit centers along x_2 , plays the role of the effective photon mass and a singular potential regularizator in Eq. (60). The lowest state corresponds to the zero value of the transverse total momentum $P_1 = 0$. In

this case Eq. (60) for the Ritus transform of the Bethe-Salpeter amplitude finally becomes

$$(-\square_2 + m^2) \Phi(t, z) = \frac{4\alpha\Phi(t, z)}{\pi(z^2 - t^2)}. \quad (62)$$

We consider now the consequences of the fall-down onto the center phenomenon present in Eq. (62), formally valid for an infinite magnetic field, and the alterations introduced by its finiteness.

A. Fall-down onto the center in the Bethe-Salpeter amplitude for strong magnetic field

In the most symmetrical case, when the wave function $\Phi(x) = \Phi(s)$ does not depend on the hyperbolic angle ϕ in the space-like region of the two-dimensional Minkowsky space, $t = s \sinh \phi$, $z = s \cosh \phi$, $s = \sqrt{z^2 - t^2}$, Eq. (62) becomes the Bessel differential equation

$$-\frac{d^2\Phi}{ds^2} - \frac{1}{s} \frac{d\Phi}{ds} + m^2\Phi = \frac{4\alpha}{\pi s^2}\Phi. \quad (63)$$

It follows from the derivation procedure in the previous Section II that this equation is valid within the interval

$$\frac{1}{\sqrt{eB}} \ll s_0 \leq s \leq \infty, \quad (64)$$

where the lower bound s_0 depends on the external magnetic field - it should be larger than the Larmour radius $L_B = (eB)^{-1/2}$ and tend to zero together with it, as the magnetic field tends to infinity. The stronger the field, the ampler the interval of validity, the closer to the origin $s = 0$ the interval of validity of this equation extends. If the magnetic field is not sufficiently strong, the lower bound s_0 falls beyond the region where the solution is mostly concentrated and the limiting form of the Bethe-Salpeter equation becomes noneffective, since it only relates to the asymptotic (large s) region, while the rest of the s -axis is served by more complicated initial Bethe-Salpeter equation, not reducible to the two-dimensional form there. This is how the strength of the magnetic field participates - note, that the coefficients of Eq. (63) do not contain it.

Solutions of (63) behave near the singular point $s = 0$ like s^σ , where

$$\sigma = \pm 2\sqrt{-\frac{\alpha}{\pi}}. \quad (65)$$

The fall-down onto the center [12] occurs, if $\alpha > \alpha_{\text{cr}} = 0$, *i.e.*, for arbitrary small attraction, the genuine value $\alpha = 1/137$ included. This differs crucially from the case of zero magnetic field where $\alpha_{\text{cr}} = \pi/8$ [13]. This difference is a purely geometrical consequence [16] of the dimensional reduction of the Minkowsky space from (1,3) to (1,1).

In discussing the physical consequences of the falling to the center we appeal to the approach recently developed by one of the present authors as applied to the Schrödinger equation with singular potential [22] and to the Dirac equation in supercritical Coulomb field [23]. Within this approach the singular center looks like a black hole. The solutions of the differential equation that oscillate near the singularity point are treated as free particles emitted and absorbed by the singularity. This treatment becomes natural after the differential equation is written as the generalized eigenvalue problem with respect to the coupling constant. Its solutions make a (rigged) Hilbert space and are subject to orthonormality relations with a singular measure. This singularity makes it possible for the oscillating solutions to be normalized to δ -functions, as free particle wave-functions should be. The nontrivial, singular measure that appears in the definition of the scalar product of quantum states in the Hilbert space of quantum mechanics introduces the geometry of a black hole of non-gravitational origin and the idea of horizon. The deviation from the standard quantum theory manifests itself in this approach only when particles are so close to one another that the mutual Coulomb field they are subjected to falls beyond the range, where the standard theory may be referred to as firmly established [23].

Following this theory we shall be using s_0 as the lower edge of the normalization box [22, 23]. For doing this it is necessary that s_0 be much smaller than the electron Compton length, $s_0 \ll m^{-1} \simeq 3.9 \times 10^{-11}$ cm, the only dimensional parameter in Eq. (63). In this case the asymptotic regime of small distances is achieved and nothing in the region $s < s_0$ beyond the normalization volume - where the two-dimensional equations (47), (52), (60), (62) and hence (63) are not valid and the space-time for charged particles remains four-dimensional - may affect the problem, because this is left behind the event horizon.

In alternative to this, we might treat s_0 as the cut-off parameter. In this case we have had to extend Eq. (63) continuously to the region $0 \leq s \leq s_0$, simultaneously replacing the singularity s^{-2} in it by a model function of s , nonsingular in the origin, say, a constant s_0^{-2} . In this approach the results are dependent on the choice of the model function which is intended to substitute for the lack of a treatable equation in that region. Besides, the

limit $s_0 \rightarrow 0$ does not exist. The latter fact implies that the approach should become invalid for sufficiently small s_0 , i.e., large B . We, nevertheless, shall also test the consequences of this approach later in this section to make sure that in our special problem the result is not affected any essentially.

B. Ultimate magnetic field

With the substitution $\Phi(s) = \Psi(s)/\sqrt{s}$ Eq. (63) acquires the standard form of a Schrödinger equation

$$-\frac{d^2\Psi(s)}{ds^2} + \frac{-4\frac{\alpha}{\pi} - \frac{1}{4}}{s^2}\Psi(s) + m^2\Psi(s) = 0. \quad (66)$$

Equation (66) is valid in the interval

$$s_0 \leq s \leq \infty, \quad s_0 \gg L_B = (eB)^{-1/2} \quad (67)$$

Treating the applicability boundary s_0 of this equation as the lower edge of the normalization box, as discussed above, $s_0 \ll m^{-1}$, we impose the standing wave boundary condition

$$\Psi(s_0) = 0, \quad (68)$$

on the solution of (66)

$$\Psi(s) = \sqrt{s} \mathcal{K}_\nu(ms), \quad \nu = i2\sqrt{\alpha/\pi} \simeq 0.096 i \quad (69)$$

that decreases at infinity. It behaves near the singular point $s = 0$ as

$$\left(\frac{s}{2}\right)^{1+\nu} \frac{1}{\Gamma(1+\nu)} - \left(\frac{s}{2}\right)^{1-\nu} \frac{1}{\Gamma(1-\nu)}. \quad (70)$$

Here the Euler Γ -functions appear. Starting with a certain small value of the argument ms , the McDonald function with imaginary index $\mathcal{K}_\nu(ms)$ (69) oscillates, as $s \rightarrow 0$, passing the zero value infinitely many times. Therefore, if s_0 is sufficiently small the standing wave boundary condition (68), prescribed by the theory of Refs. [22, 23], can be definitely satisfied. Keeping to the genuine value of the coupling constant $\alpha = 1/137$ ($\nu = 0.096 i$) one may ask: what is the largest possible value s_0^{\max} of s_0 , for which the boundary problem (66), (68) can be solved? By demanding, in accord with the validity condition (64) of Eqs. (63) and (66), that the value of s_0^{\max} should exceed the Larmour radius:

$$s_0^{\max} \gg (eB)^{-1/2} \quad \text{or} \quad B \gg \frac{1}{e (s_0^{\max})^2}, \quad (71)$$

one establishes, how large the magnetic field should be in order that the boundary problem might have a solution, in other words, that the point $P_0 = \mathbf{P} = 0$ might belong to the spectrum of bound states of the Bethe-Salpeter equation in its initial form (3).

One can use the asymptotic form of the McDonald function near zero to see that the boundary condition (68) is satisfied provided that

$$\left(\frac{ms_0}{2}\right)^{2\nu} = \frac{\Gamma(1+\nu)}{\Gamma^*(1-\nu)} \quad (72)$$

or

$$\nu \ln \frac{ms_0}{2} = i \arg \Gamma(\nu + 1) - i\pi n, \quad n = 0, \pm 1, \pm 2, \dots \quad (73)$$

Since $|\nu|$ is small we may exploit the approximation $\Gamma(1+\nu) \simeq 1 - \nu C_E$, where $C_E = 0.577$ is the Euler constant, to get

$$\ln \left(\frac{ms_0}{2}\right) = -\frac{n}{2} \sqrt{\frac{\pi^3}{\alpha}} - C_E, \quad n = 1, 2, \dots \quad (74)$$

We have expelled the non-positive integers n from here, since they would lead to the roots for ms_0 of the order of or larger than unity in contradiction to the adopted condition $s_0 \ll m^{-1}$. For such values eq. (70) is not valid. It may be checked that there are no other zeros of McDonald function, besides (74). The maximum value for s_0 is provided by $n = 1$. We finally get

$$\begin{aligned} \ln \left(\frac{ms_0^{\max}}{2}\right) &= -\frac{1}{2} \sqrt{\frac{\pi^3}{\alpha}} - C_E \\ &\text{or} \\ s_0^{\max} &= \frac{2}{m} \exp \left\{ -\frac{1}{2} \sqrt{\frac{\pi^3}{\alpha}} - C_E \right\} \simeq 10^{-14} \frac{1}{m}. \end{aligned} \quad (75)$$

This is fourteen orders of magnitude smaller than the Compton length $m^{-1} = 3.9 \times 10^{-11} \text{ cm}$ and makes about 10^{-25} cm . Now, in accord with (71), if the magnetic field exceeds the ultimate value of

$$B_{\text{ult}} = \frac{m^2}{4e} \exp \left\{ \frac{\pi^{3/2}}{\sqrt{\alpha}} + 2C_E \right\} \simeq 1.6 \times 10^{28} B_0, \quad (76)$$

the positronium ground state with the center-of-mass 4-momentum equal to zero appears. Here $B_0 = m^2/e \simeq 1.22 \times 10^{13}$ Heaviside-Lorentz units is the Schwinger critical field, or $B_0 = m^2 c^3 / e \hbar \simeq 4.4 \times 10^{13} \text{ G}$. The value of B_{ult} is $\sim 10^{42} \text{ G}$ that is a few orders of

magnitude smaller than the highest magnetic field in the vicinity of superconductive cosmic strings [7]. Excited positronium states may also reach the spectral point $P_\mu = 0$, but this occurs for magnetic fields, tens orders of magnitude larger than (76) - to be found in the same way from (74) with $n = 2, 3...$

The ultra-relativistic state $P_\mu = 0$ has the internal structure of what was called a *confined* state in [22, 23], i.e. the one whose wave function behaves as a standing wave combination of free particles incoming from behind the lower edge of the normalization box and then totally reflected back to this edge. It decreases as $\exp(-ms)$ at large distances like the wave function of a bound state. The effective "Bohr radius", i.e. the value of s that provides the maximum to the wave function (69) makes $s_{\max} = 0.17m^{-1}$ (this fact is established by numerical analysis). This is certainly much less than the standard Bohr radius $a_0 = (\alpha m)^{-1}$. Taken at the level of 1/2 of its maximum value, the wave-function is concentrated within the limits $0.006 m^{-1} < s < 1.1 m^{-1}$. But the effective region occupied by the confined state is still much closer to $s = 0$. The point is that the probability density of the confined state is the wave function squared *weighted with the measure $s^{-2}ds$ singular in the origin* [22], [23] and is hence concentrated near the edge of the normalization box $s_0 \simeq 10^{-25}$ cm, and not in the vicinity of the maximum of the wave function. The electric fields at such distances are about 10^{43} Volt/cm. Certainly, there is no evidence that the standard quantum theory should be valid under such conditions. This remark gives the freedom of applying the theory presented in Refs. [22], [23].

A relation like (76) between a Fermion mass and the magnetic field is present in [24]. There, however, a different problem is studied and, correspondingly, a different meaning is attributed to that relation: it expresses the mass acquired dynamically by a primarily massless Fermion in terms of the magnetic field applied to it. The mass generation is described by the homogeneous Bethe-Salpeter equation, whose solution is understood [14, 24] as the wave function of the Goldstone boson corresponding to the spontaneous breaking of the chiral symmetry characteristic of the massless QED. It is claimed, moreover, that the resulting relation between the magnetic field and the acquired mass is independent of the choice of the gauge for the photon propagator. The equations of Ref. [24] may well be read off, formally, as serving our problem of the compensation of the positronium rest mass by the mass defect in a magnetic field, too, and the resulting expression may be used for determining the corresponding magnetic field, provided that the electron mass m is substituted for the

acquired mass m_{dyn} of [24]. There is, however, an important discrepancy in numerical coefficients in the characteristic exponential between (76) and the corresponding formula in [24]: the latter contains $\exp\{\pi^{3/2}/(2\alpha)^{1/2}\}$ in place of $\exp\{\pi^{3/2}/\alpha^{1/2} + 2C_E\}$ in (76) and its direct use would lead to a more favorable estimate of the ultimate value of the magnetic field, $2.6 \times 10^{19} B_0$, than (76). Although the basic mechanisms, the dimensional reduction and falling to the center, acting here and in [24], are essentially the same, the procedures are very much different, and the origin of the discrepancy remains unclear. Later, in [25] the authors revised their relation in favor of a different approximation. Supposedly, the revised relation may be also of use in the problem of ultimate magnetic field dealt with here.

It is interesting to compare the value (76) with the analogous value, obtained earlier by the present authors (see p.393 of Ref. [10]) by extrapolating the nonrelativistic result concerning the positronium binding energy in a magnetic field to extreme relativistic region:

$$B_{\text{ult}}|_{\text{NON-REL}} = \frac{\alpha^2 m^2}{e} \exp\left\{\frac{2\sqrt{2}}{\alpha}\right\} \simeq 10^{164} B_0. \quad (77)$$

Such is the magnetic field that makes the binding energy of the lowest energy state equal to $(-2m)$. (This is worth comparing with the magnetic field, estimated [26] as $\alpha^2 \exp(2/\alpha) B_0$, that makes the mass defect of the nonrelativistic hydrogen atom comparable with the electron rest mass). We see that the relativistically enhanced attraction has resulted in a drastically lower value of the ultimate magnetic field. Note the difference in the character of the essential nonanalyticity with respect to the coupling constant: it is $\exp(\pi\sqrt{\pi}/\sqrt{\alpha})$ in (76) and $\exp(2\sqrt{2}/\alpha)$ in (77). Another effect of relativistic enhancement is that within the semi-relativistic treatment of the Bethe-Salpeter equation [9]-[11], as well as within the one using the Schrödinger equation [1], only the lowest level could acquire unlimited negative energy with the growth of the magnetic field, whereas according to (74) in our fully relativistic treatment all excited levels with $n > 1$ are subjected to the falling to the center and can reach in turn the point $P_{\parallel} = 0$.

Let us see now, how the result (76) is altered if the cut-off procedure of Ref. [12] is used. Consider Eq. (66) in the domain $s_0 < s < \infty$, but replace it with another equation

$$-\frac{d^2\Psi_0(s)}{ds^2} - \frac{\frac{4\alpha}{\pi} + \frac{1}{4}}{s_0^2}\Psi_0(s) + m^2\Psi_0(s) = 0 \quad (78)$$

in the domain $0 < s < s_0$. The singular potential is replaced by a constant near the origin in (78). Demand, in place of (68), that $\Psi_0(0) = 0$, $[\Psi'_0(s_0)/\Psi_0(s_0)] = (\Psi'(s_0)/\Psi(s_0))$. Then,

the result (76) will be modified by the factor

$$\exp \left\{ -\frac{2}{\sqrt{\frac{4\alpha}{\pi} + \frac{1}{4}} \cot \left(\frac{4\alpha}{\pi} + \frac{1}{4} \right) - \frac{1}{2}} \right\}, \quad (79)$$

which may be taken at the value $\alpha = 0$. Thus, the result (76) is only modified by a factor of $\exp(-4/3) \simeq 0.25$. Generally, the estimate of the limiting magnetic field (76) is practically nonsensitive to the way of cut-off, in other words to any solution of the initial equation inside the region $0 < s < s_0$, where the magnetic field does not dominate over the mutual attraction force between the electron and positron. This fact takes place, because the term $(\pi^{3/2}/\sqrt{\alpha}) \simeq 65$, singular in α , is prevailing in (76), the details of the behavior of the wave function close to the origin $s = 0$ being not essential against its background.

C. Radiative corrections

1. Vacuum polarization

We should answer the question of whether the effects of vacuum polarization in a strong magnetic field may or may not screen the interaction between the electron and positron in such a way as to prevent the falling to the center in the positronium atom. It is clear *a posteriori* that no matter how strong the magnetic field is, the ultraviolet singularity dominates over its influence in the photon propagator, if the interval sufficiently close to the light cone is involved. Therefore, there is a competition between the magnetic field and this characteristic interval, which is in our problem the Larmour radius that itself depends on the magnetic field. We have to consider the outcome of this competition.

To include the effect of the vacuum polarization we should use the photon propagator in a magnetic field, whose influence is realized via the vacuum polarization radiative corrections, instead of its free form (24) used above. The photon propagator in a constant and homogeneous magnetic field has the following approximation-independent structure [27]-[29]

$$D_{ij}(x) = \frac{1}{(2\pi)^4} \int \exp(ikx) D_{ij}(k) d^4k, \quad i, j = 0, 1, 2, 3, \quad (80)$$

$$D_{ij}(k) = \sum_{a=1}^4 D_a(k) \frac{b_i^{(a)} b_j^{(a)}}{(b^{(a)})^2},$$

$$D_a(k) = \begin{cases} -[k^2 + \kappa_a(k)]^{-1}, & a = 1, 2, 3 \\ \text{arbitrary}, & a=4 \end{cases}. \quad (81)$$

Here $b^{(a)}$ and κ^a are four eigenvectors and four eigenvalues of the polarization operator Π_{ij}

$$\Pi_i^j b_j^{(a)} = \kappa_a(k) b_i^{(a)}. \quad (82)$$

The eigenvectors are known in the final form:

$$b_i^{(1)} = (F^2 k)_i k^2 - k_i (k F^2 k), \quad b_i^{(2)} = (\tilde{F} k)_i, \quad b_i^{(3)} = (F k)_i, \quad b_i^{(4)} = k_i, \quad (83)$$

where F , \tilde{F} and F^2 are the external electromagnetic field tensor, its dual, and its tensor squared, respectively, contracted with the photon 4-momentum k . On the contrary, the eigenvalues $\kappa_{1,2,3}(k)$ are generally unknown - subject to approximate calculations - scalar functions of two Lorentz-invariant combinations of the momentum and the field, which in the special frame, where the external electromagnetic field is given by (7), are $k_0^2 - k_3^2$ and $k_1^2 + k_2^2 \equiv k_\perp^2$. The eigenvalue κ_4 is equal to zero as a trivial consequence of the transversality $\Pi_i^j k_j = 0$ of the polarization operator. The eigenvectors (83) with $a = 1, 2, 3$ are 4-potentials of the three photon modes, while the dispersion laws of the corresponding electromagnetic eigenwaves are obtained by equalizing the denominators in (81) with zero. In the special frame the eigenvectors (83), up to normalizations, are

$$b_i^{(1)} = k^2 \begin{pmatrix} 0 \\ k_1 \\ k_2 \\ 0 \end{pmatrix}_i + k_\perp^2 \begin{pmatrix} k_0 \\ k_1 \\ k_2 \\ k_3 \end{pmatrix}_i, \quad b_i^{(2)} = \begin{pmatrix} k_3 \\ 0 \\ 0 \\ k_0 \end{pmatrix}_i, \quad b_i^{(3)} = \begin{pmatrix} 0 \\ k_2 \\ -k_1 \\ 0 \end{pmatrix}_i. \quad (84)$$

When calculated [27], [28] within the one-loop approximation of the Furry picture (i.e. using exact Dirac propagators in the external magnetic field without radiative corrections) these eigenvalues have the following asymptotic behavior [30], [28], [29], [31], [32] (note the difference in the signs in front of k^2 due to a different metric convention used here) for large fields $eB \gg m^2$, $eB \gg |k_3^2 - k_0^2|$

$$\kappa_1(k_0^2 - k_3^2, k_\perp^2) = \frac{-\alpha k^2}{3\pi} \left(\ln \frac{B}{B_0} - C - 1.21 \right), \quad (85)$$

$$\begin{aligned}
& \kappa_2(k_0^2 - k_3^2, k_\perp^2) = \\
& = \frac{\alpha B m^2 (k_0^2 - k_3^2)}{\pi B_0} \exp \left(-\frac{k_\perp^2}{2m^2} \frac{B_0}{B} \right) \int_{-1}^1 \frac{(1 - \eta^2) d\eta}{4m^2 - (k_0^2 - k_3^2)(1 - \eta^2)},
\end{aligned} \tag{86}$$

$$\begin{aligned}
& \kappa_3(k_0^2 - k_3^2, k_\perp^2) = \\
& = \frac{-\alpha k_\perp^2}{3\pi} \left(\ln \frac{B}{B_0} - C \right) - \frac{\alpha}{3\pi} [0.21 k_\perp^2 - 1.21 (k_0^2 - k_3^2)].
\end{aligned} \tag{87}$$

$C = 0.577$ is the Euler constant. Eqs. (85) and (87) are accurate up to terms, decreasing with B like $(B_{cr}/B) \ln(B/B_{cr})$ and faster. Eq. (86) is accurate up to terms, logarithmically growing with B . In $\kappa_{1,3}$ we took also the limit $k_\perp^2 \ll (B/B_{cr})m^2$, which is not the case for κ_2 , wherein the factor $\exp(-k_\perp^2 B_0/2m^2 B)$ is kept different from unity. Although the components $\kappa_{1,2,3}$ contain the growing logarithms $\alpha \ln(B/B_0)$ the latter are yet small for the values of the magnetic field of the order of B_{ult} (76). This is not the case for the linearly growing part of (86).

Let us inspect the contributions of the photon propagator (81) into the equation that should appear in place of (47). To match the diagonal form (24) corresponding to the Feynman gauge we fix the gauge arbitrariness by choosing

$$D_4(k) = -[k^2 + \kappa_1(k)]^{-1}. \tag{88}$$

In the isotropic case where no magnetic field is present all the three nontrivial eigenvalues are the same, $\kappa_a(k) = \kappa(k)$, $a = 1, 2, 3$. Then, with the choice (88) in (81) the photon propagator in this limit becomes diagonal

$$D_{ij}(k) = -\frac{1}{k^2 + \kappa(k)} \sum_{a=1}^4 \frac{b_i^{(a)} b_j^{(a)}}{(b^{(a)})^2} = -\frac{g_{ij}}{k^2 + \kappa(k)}, \tag{89}$$

since the eigenvectors (82) or (84) make an orthogonal basis irrespective of whether the magnetic field is present or not.

In spite of the presence [17], [30] of a term, linearly growing with the field in (86), the component D_2 does contribute in the limit of high fields into the right-hand side of an equation to replace (60), because the ultra-violet singularity at the distance of the Larmour radius from the light cone dominates. To see this note that the right-hand side of the analog of (47) should get the contribution from D_2 :

$$\frac{1}{(2\pi)^4} \int \frac{[k_3 \gamma_0 - k_0 \gamma_3]_{\lambda\lambda^e} [k_3 \gamma_0 - k_0 \gamma_3]_{\mu\lambda^p}}{(k_0^2 - k_3^2)} \frac{\exp[i(kx)]}{k^2 + \kappa_2} d^4 k. \tag{90}$$

After this is contracted with the unit matrix we get for the corresponding contribution into the right-hand side of the equation to be written in place of Eq. (60) the expression

$$-\frac{1}{(2\pi)^4} \int \frac{\exp[i(kx)]}{k^2 + \kappa_2} d^4k. \quad (91)$$

Once the Hermite functions (17) restrict x_\perp in integrals like (29) and (32) to the region inside the Larmour radius, the region $k_\perp^2 \gg L_B^{-2} = m^2 B/B_0$ in the integral (91) is important. There, however, k_2 disappears due to the exponential factor in (86) and we are left with the contribution, the same as the one coming from the free photon propagator. Moreover, as the light-cone singularity is formed exclusively due to integration over near-infinite values of all the four photon momentum components, Eq. (91) behaves like $1/x^2$, the same as (24), near the light cone $x^2 = 0$.

We need, however, to also estimate the contribution immediately close to this singularity. To this end, let us disregard the spatial dispersion of the dielectric constant in the transverse plane, i.e. take κ_2 at the value $k_\perp = 0$ in (91). By doing so we essentially underestimate the contribution of the mode-2 photon as a carrier of the electromagnetic interaction into the attraction force between the electron and positron near the light cone, because we keep the term linearly growing with the field in the denominator for large k_\perp , where it in fact disappears. This approximation does not affect the light-cone singularity, which remains $1/x^2$, but makes the screening correction to the singular part larger than it is. We shall see, nevertheless, that even within this approximation, with the screening overestimated, the effect of the latter is small. Once our working domain is restricted to the intervals $z^2 - t^2$ much closer to the light cone than the Compton length, we may keep to the condition $|k_0^2 - k_3^2| \gg m^2$ in the integral (91). Then κ_2 (86) should be taken in (91) as

$$\kappa_2 = -\frac{2\alpha B m^2}{\pi B_0}. \quad (92)$$

Then (91) becomes the well-known expression for the free propagator of a massive particle with the mass squared $M^2 = -\kappa_2$. (To avoid a possible misunderstanding, stress that this mass should not be referred to as an effective photon mass. The mass of the photon defined as its rest energy is always equal to zero, corresponding to the fact that the point $k_0 = k_3 = k_1 = k_2 = 0$ is solution to the dispersion equations $k^2 + \kappa_a(k) = 0$. The "mass" M appears in the ultraviolet, and not infrared regime.) In the adiabatic approximation of Subsection C of Section II the dimensional reduction yields again the prescription to

disregard the dependence on x_\perp in it by setting $x_\perp = 0$, $x^2 = -s^2$. Then the contribution from (91) is

$$\frac{iM}{4\pi^2 s} \mathcal{K}_1(Ms) \simeq \frac{i}{4\pi^2 s^2} \left(1 - \frac{s^2 M^2}{2} \left| \ln \frac{Ms}{2} \right| \right). \quad (93)$$

Here \mathcal{K}_1 is the McDonald function of order one, and we have pointed its asymptotic behavior near the point $s^2 = z^2 - t^2 = +0$. According to (75) near the lower edge of the normalization volume and with the magnetic field (76) the quantity $sM = (2\alpha/\pi)^{1/2}$ makes 0.068, and hence the second term inside the brackets in (93) is only -7.8×10^{-3} . Therefore the screening effect, although overestimated, is still negligible, the contribution of D_2 making one half of the full contribution of the free photon propagator considered above. The one half originates from the absence of the factor 2 that appeared above when $[\gamma_i]_{\lambda\lambda^e}[\gamma_i]_{\mu\lambda^p}$ in (47) was later contracted with the unity I to lead to (60): $\sum_{i=0,3} g_{ii}\gamma_i\gamma_i = 2$. The other half comes from the contribution of D_1 and D_4 .

The quantity D_3 contains only k_\perp components that give rise to $[k^i\gamma_i]_{\lambda\lambda^e}[k^j\gamma_j]_{\mu\lambda^p}$, $i, j = 1, 2$, in an equation to appear in place of (32), and consequently contribute only to the nondiagonal in Landau quantum numbers part of the Bethe-Salpeter equation like (40), that does not survive in the limit of high magnetic field. On the contrary, the contributions of D_1 and D_4 go to the diagonal part. This occurs because these contain the components k_0 and k_3 carrying the matrices γ_0 and γ_3 that may lead to the term diagonal in Landau quantum numbers, as explained when passing from (39) to (46) and (47). It follows from (81), (84), (88) that the common contribution from D_1 and D_4 in the (0,3) subspace is determined by the expression

$$\frac{(k_\perp^2)^2 k_i k_j}{(b^{(1)})^2} + \frac{k_i k_j}{k^2} = \frac{k_i k_j}{k_0^2 - k_3^2}, \quad i, j = 0, 3. \quad (94)$$

Then the counterpart of (90) reads

$$- \frac{1}{(2\pi)^4} \int \frac{[k_0\gamma_0 - k_3\gamma_3]_{\lambda\lambda^e} [k_0\gamma_0 - k_3\gamma_3]_{\mu\lambda^p}}{(k_0^2 - k_3^2)} \frac{\exp[i(kx)]}{k^2 + \kappa_1} d^4k, \quad (95)$$

and the counterpart of (91) becomes (again with the disregard of the spatial dispersion across the magnetic field already done when writing Eq. (85))

$$- \frac{1}{\mathcal{A}(2\pi)^4} \int \frac{\exp[i(kx)]}{k^2} d^4k = \frac{1}{\mathcal{A}i4\pi^2 x^2}, \quad (96)$$

where

$$\mathcal{A} = 1 - \frac{\alpha}{3\pi} \left(\ln \frac{B}{B_0} - C - 1.21 \right) \quad (97)$$

in view of (85). For the fields as large as $B = B_{\text{ult}}$ (76) the number \mathcal{A} is very close to unity: $\mathcal{A} = 1 - 0.04$. (Its difference from unity is the measure of the anti-screening effect of the running coupling constant α/\mathcal{A} for large magnetic field due to the lack of asymptotic freedom in pure quantum electrodynamics).

We conclude that the vacuum polarization does not any essentially affect the falling to the center and hence the estimate of the ultimate magnetic field. This contradicts the prescription to replace $\alpha \rightarrow \alpha/2$ in the expression for the latter that would result if one applied the corresponding conclusion from Ref. [24] to the problem under consideration. The point is that in Ref. [24] the contribution of D_2 is completely disregarded for the reason that the term (92), linearly growing with the magnetic field, is in the denominator of D_2 . We saw above that that this cannot be done: it essentially contributes to the falling to the center asymptotic regime of $s \gg 10^{-11}m^{-1}$, where the probability to find the system is concentrated.

Gathering the results of the present consideration together we conclude that the effect of the vacuum polarization leads, in the approximation where the spatial dispersion in the orthogonal direction is neglected, to the replacement of Eq. (58) by the following two-dimensional Bethe-Salpeter equation for high magnetic field limit including an external electric field and the effects of the vacuum polarization

$$\begin{aligned} & \left[i\hat{\partial}_{\parallel}^e - e\hat{A}_{\parallel}(x_{\parallel}^e) - m \right]_{\lambda\beta} \left[i\hat{\partial}_{\parallel}^p + e\hat{A}_{\parallel}(x_{\parallel}^p) - m \right]_{\mu\nu} \left[\chi_{0,p_1^e;0,p_1^p}(x_{\parallel}^e, x_{\parallel}^p) \right]_{\beta\nu} \\ &= \frac{\alpha}{\pi} \left\{ \frac{[i\hat{\partial}_{\parallel}]_{\lambda\beta} [i\hat{\partial}_{\parallel}]_{\mu\nu}}{\mathcal{A}\square_2} \frac{1}{z^2 + \frac{P_1^2}{(eB)^2} - t^2} + \right. \\ & \left. \frac{[\gamma_0 i\partial_z + \gamma_3 i\partial_t]_{\lambda\beta} [\gamma_0 i\partial_z + \gamma_3 i\partial_t]_{\mu\nu}}{\square_2} \frac{MK_1 \left(M(z^2 + \frac{P_1^2}{(eB)^2} - t^2)^{\frac{1}{2}} \right)}{(z^2 + \frac{P_1^2}{(eB)^2} - t^2)^{\frac{1}{2}}} \right\} \left[\chi_{0,p_1^e;0,p_1^p}(x_{\parallel}^e, x_{\parallel}^p) \right]_{\beta\nu}, \quad (98) \end{aligned}$$

Here the action of the derivatives over t and z does not extend beyond the braces, $i\hat{\partial}_{\parallel} = i\gamma_0\partial_t + i\gamma_3\partial_z$, $\square_2 = \partial_z^2 - \partial_t^2$. Remind that $t = x_0^e - x_0^p$, $z = x_3^e - x_3^p$. For no electric field case the equation that follows from (98) for the singlet component to substitute for (60) is

$$(-\square_2 + m^2) \Phi(t, z) = \frac{2\alpha}{\pi} \left\{ \frac{1}{\mathcal{A}(z^2 + \frac{P_1^2}{(eB)^2} - t^2)} + \frac{MK_1 \left(M(z^2 + \frac{P_1^2}{(eB)^2} - t^2)^{\frac{1}{2}} \right)}{(z^2 + \frac{P_1^2}{(eB)^2} - t^2)^{\frac{1}{2}}} \right\} \Phi(t, z) \quad (99)$$

Finally, the Bessel equation (63) for the (1,1) rotationally invariant solution now becomes

$$-\frac{d^2\Phi}{ds^2} - \frac{1}{s} \frac{d\Phi}{ds} + m^2\Phi = \frac{2\alpha}{\pi s} \left(\frac{1}{s} + MK_1(Ms) \right) \Phi. \quad (100)$$

We neglected the difference of \mathcal{A} from unity.

2. Mass corrections

Mass radiative corrections should be taken into account by inserting the mass operator into the Dirac differential operators in the l.h. sides of the Bethe-Salpeter equation (3) or (47). We shall estimate now, whether this may affect the above conclusions concerning the positronium mass compensation by the mass defect.

In strong magnetic field the one-loop calculation of the electron mass operator leads to the so-called double-logarithm mass correction growing with the field B as [33]

$$\tilde{m} = m \left(1 + \frac{\alpha}{4\pi} \ln^2 \frac{B}{B_0} \right). \quad (101)$$

For $B \simeq B_{\text{ult}}$ the corrected mass makes $\tilde{m} = 3.45m$. This implies that the mass annihilation due to the falling to the center is opposed by the radiative corrections and requires a field somewhat larger than (76). To determine its value, substitute \tilde{m} (101) for m and $L_B = (eB)^{-1/2}$ for s_0 into equation (74) with $n = 1$. The resulting equation for the ultimate magnetic field, modified by the mass radiative corrections, B_{corr} ,

$$\left(1 + \frac{\alpha}{4\pi} \ln^2 \frac{B_{\text{corr}}}{B_0} \right)^2 = 4 \frac{B_{\text{corr}}}{B_0} \exp \left(-\sqrt{\frac{\pi^3}{\alpha}} + C_E \right), \quad (102)$$

has the numerical solution: $B_{\text{corr}} \simeq 13 B_{\text{ult}}$.

When going beyond the one-loop approximation by summing the rainbow diagrams two different expressions for \tilde{m} were obtained by different authors. Ref. [34] reports

$$\tilde{m} = m \exp \left(\frac{\alpha}{4\pi} \ln^2 \frac{B}{B_0} \right). \quad (103)$$

The use of this formula analogous to the above gives rise to an increase of the ultimate value by two orders of magnitude: $B_{\text{corr}} = 3.5 \times 10^2 B_{\text{ult}}$, whereas the use of the result of Ref. [35]

$$\tilde{m} = \frac{m}{\cos \left(\sqrt{\frac{\alpha}{2\pi}} \ln \frac{B}{B_0} \right)} \quad (104)$$

would leave the ultimate value practically unchanged: $B_{\text{corr}} = 1.5 B_{\text{ult}}$. Finally, if the vacuum polarization is taken into account while summing the leading contributions to the large field asymptotic behavior of the mass operator, the following result [36]

$$\tilde{m} = \frac{m}{1 - \frac{\alpha}{2\pi} \left(\ln \frac{\pi}{\alpha} - C_E \right) \ln \frac{B}{B_0}} \quad (105)$$

was obtained, from where the double logarithm is absent due to the effect of the term (92) in the photon propagator when substituted into electron-photon loops. The use of (105) would result in: $B_{\text{corr}} = 3B_{\text{ult}}$.

Anyway, we see that the mass correction, increasing the ultimate value B_{ult} by at the most two orders of magnitude, is not essential bearing in mind the huge values (76) of the latter. Moreover, basing on the most recent results concerning the mass correction [36] we conclude that the latter do not affect the value of the hypercritical field obtained above (76) practically at all.

IV. SUMMARY AND DISCUSSION

In the paper we have considered the system of two charged relativistic particles - especially the electron and positron - in interaction with each other, when placed in a strong constant and homogeneous magnetic field \mathbf{B} . The Bethe-Salpeter equation in the ladder approximation in the Feynman gauge is used without exploiting any non-relativistic assumption. We have derived the ultimate two-dimensional form of the Bethe-Salpeter equation, when the magnetic field tends to infinity, with the help of expansion over the complete set of Ritus matrix eigenfunctions [19]. The latter accumulate the spacial and spinor dependence on the transversal-to-the-field degree of freedom. The Fourier-Ritus transform of the Bethe-Salpeter amplitude obeys an infinite chain of coupled differential equations that decouple in the limit of large B , so that we are left with one closed equation for the amplitude component with the Landau quantum numbers of the electron and positron both equal to zero, while the components with other values of Landau quantum numbers vanish in this limit. The resulting equation is a differential equation with respect to two variables that are the differences of the particle coordinates: along the time $t = x_0^e - x_0^p$ and along the magnetic field $z = x_3^e - x_3^p$. It contains only two Dirac matrices γ_0 and γ_3 and can be alternatively written using 2×2 Pauli matrices. The term responsible for interaction with a moderate electric field \mathbf{E} directed along \mathbf{B} , $E \ll B$, is also included and does not lay obstacles to the dimensional reduction. By introducing different masses the resulting two-dimensional equation may be easily modified to cover also the case of an one-electron atom in strong magnetic field and/or other pairs of charged particles.

It is worth noting that the two-dimensionality holds only with respect to the degrees of

freedom of charged particles, while the photons remain 4-dimensional in the sense that the singularity of the photon propagator is determined by the inverse d'Alembert operator in the 4-dimensional, and not two-dimensional Minkowsky space. (Otherwise it would be weaker).

We have made sure that in the case under consideration the critical value of the coupling constant is zero, $\alpha_{\text{cr}} = 0$, i.e., the falling to the center caused by the ultraviolet singularity of the photon propagator as a carrier of the interaction is present already for its genuine value $\alpha = 1/137$, in contrast to the no-magnetic-field case, where $\alpha_{\text{cr}} > 1/137$. If the magnetic field is large, but finite, the dimensional reduction holds everywhere except a small neighborhood of the singular point $s = 0$, wherein the mutual interaction between the particles dominates over their interaction with the magnetic field. The dimensionality of the space-time in this neighborhood remains to be 4, and its size is determined by the Larmour radius $L_B = (eB)^{-1/2}$ that is zero in the limit $B = \infty$. The latter supplies the singular problem with a regularizing length. The larger the magnetic field, the smaller the regularizing length, and the deeper the level.

We have found the ultimate magnetic field that provides the full compensation of the positronium rest mass by the binding energy, and the wave function of the corresponding state as a solution to the Bethe-Salpeter equation. This state is described in terms of the theory of the falling to the center, developed in [22, 23], as a "confined" state, different from the usual bound state. The appeal to this theory is necessitated by the fact that the falling to the center draws the electron and positron so close together that the mutual field is so large that the standard treatment may become inadequate. The ultimate value is estimated to be unaffected by the radiative corrections modifying the mass and polarization operators.

In spite of the huge value, expected to be present, perhaps, only in superconducting cosmic strings [7], the magnetic field magnitude obtained may be important as setting the limits of applicability of QED or presenting the ultimate value of the magnetic field admissible within pure QED. The point is that at this field the energy gap separating the electron-positron system from the vacuum disappears. An exceeding of the ultimate magnetic field would cause restructuring of the vacuum. The question about the vacuum restructuring typical of other problems - with or without the magnetic field, where the falling-to-the-center takes place: the supercharged nucleus [37, 38] and a moderately charged nucleus with strong magnetic field [39], is discussed in the two adjacent papers [16, 40]. The formal mechanisms that realize the magnetic field instability and may lead to prevention of its further growth via

the decay of the "confined" state found here require a further study and will be considered elsewhere.

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